9th Week Wednesday

19 Nisan 2023 Çarşamba 13:49

Typical vector

$V = \mathbb{R}^3 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $a, b, c \in \mathbb{R}$

$V = \mathbb{R}^2 \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}$ $a, b \in \mathbb{R}$

$V = \mathbb{R}^{2\times 2} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a, b, c, d \in \mathbb{R}$

$V = \mathbb{P}_3 \Rightarrow ax^2 + bx + c$ $a, b, c \in \mathbb{R}$

Is $\{v_1, v_2, \dots, v_n\}$ a spanning set for V?

$\Rightarrow$ $\text{Span}\{v_1, v_2, \dots, v_n\} \stackrel{?}{=} V$

$\Rightarrow$ Does $\{v_1, v_2, \dots, v_n\}$ span V?

You should be able to write any vector in V → a typical vector in V

as a linear combination of $\{v_1, v_2, \dots, v_n\}$.

$r_1 v_1 + r_2 v_2 + \dots + r_n v_n =$ typical vector of V

If there is a solution for $r_1, r_2, \dots, r_n$ for all ✓

(we don't want no soln case possible!) $\Rightarrow \{v_1, v_2, \dots, v_n\}$ spans V.

Ex $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ Does $\{v_1, v_2\}$ span $\mathbb{R}^3$? → a typical element $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$r_1 v_1 + r_2 v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ → Can we find a solution for r_1, r_2 for all a, b, c without any restrictions?

$r_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + r_2 \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\begin{aligned} r_1 + 2r_2 &= a \\ 2r_1 + 0r_2 &= b \\ 3r_1 + 5r_2 &= c \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 2 & 0 & b \\ 3 & 5 & c \end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -4 & b-2a \\ 0 & -1 & c-3a \end{array}\right]$$

$$\xrightarrow[\text{then } -1r_2\to r_2]{r_2 \leftrightarrow r_3} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & 3a-c \\ 0 & -4 & b-2a \end{array}\right] \xrightarrow{4r_2+r_3\to r_3} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & 3a-c \\ 0 & 0 & 10a+b-4c \end{array}\right]$$

$12a - 4c + b - 2a$

If $10a + b - 4c \neq 0$ then the system has no soln.

gives a restriction for a, b, c

gives a restriction for a,b,c } then the system has no soln.

$\Rightarrow \{v_1, v_2\}$ does not span $\mathbb{R}^3$.

Ex

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$?

Yes ✓

$$r_1 v_1 + r_2 v_2 + r_3 v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{array}{ccc|c} r_1 & r_2 & r_3 & \\ \end{array}$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & -1 & 1 & a \\ 0 & 1 & 2 & b \\ 2 & 3 & 3 & c \end{array}\right]}_{A} \rightarrow - - -$$

square system $\rightarrow$ $\det(A) \neq 0 \Rightarrow$ there is a solution. ✓

$\det(A) = 0 \Rightarrow$ no soln.

$$\det(A) = 1 \underbrace{\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}}_{-3} - (-1) \underbrace{\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}}_{-4} + 1 \cdot \underbrace{\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}}_{-2} = -9 \neq 0$$

Ex

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$?

Not a spanning set ✓.

$$r_1 v_1 + r_2 v_2 + r_3 v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 2 & -1 & 3 & b \\ 3 & 2 & 1 & c \end{array}\right]$$

$$\det(A) = 1 \underbrace{\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}}_{-1-6=-7} - 0 + 1 \underbrace{\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}}_{4-(-3) = 7} = -7 + 7 = 0$$

$\Rightarrow$ No soln. case possible.

Ex

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Does $\{v_1, v_2, v_3, v_4\}$ span $\mathbb{R}^3$?

✓ Yes

$$r_1 v_1 + r_2 v_2 + r_3 v_3 + r_4 v_4 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & a \\ 2 & 1 & 2 & 3 & b \\ 3 & 2 & 0 & 4 & c \end{array}\right] \xrightarrow[-3r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & a \\ 0 & 1 & 4 & 1 & b-2a \\ 0 & 2 & 3 & 1 & c-3a \end{array}\right]$$

$$\xrightarrow{-2r_2 + r_3 \to r_3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & a \\ 0 & 1 & 4 & 1 & b-2a \\ 0 & 0 & -5 & -1 & c-2b-a \end{array}\right] \xrightarrow{-\frac{1}{5} r_3 \to r_3} \begin{array}{c} \begin{array}{cccc|c} & & & r_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & a \\ 0 & 1 & 4 & 1 & b-2a \\ 0 & 0 & 1 & 1/5 & -(c-2b-a)/5 \end{array}\right] \end{array}$$

$$\longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 4 & 1 & b-2a \\ 0 & 0 & -5 & -1 & c-2b-a \end{array}\right] \xrightarrow{-\frac{1}{5}r_3 \to r_3} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 1/5 & -(c-2b-a)/5 \end{array}\right]$$

$-2b+2a$
$c-3a$

$r_4 \in \mathbb{R} \rightarrow$ inf. many soln. ✓

$\Rightarrow$ No soln case is not possible ✓

Notice that:

For $\mathbb{R}^n$: $< n$ vectors $\Rightarrow$ can not span $\mathbb{R}^n$

$\geq n$ vectors $\Rightarrow$ may or may not span $\mathbb{R}^n$ should check!

($= n$ vector $\Rightarrow$ make use of det $\Rightarrow$ det $\neq 0 \Rightarrow$ spans ✓
det $= 0 \Rightarrow$ not span X

For $\mathbb{P}_n \Rightarrow$ //

For $\mathbb{R}^{m\times n} \Rightarrow$ $< m.n$ vectors $\Rightarrow$ can not span $\mathbb{R}^{m\times n}$

$\geq mn$ vectors $\Rightarrow$ may or may not span $\mathbb{R}^{m\times n}$

Ex/ $\mathbb{P}_3$) $\Rightarrow$ $v_1 = 1+x$, $v_2 = x^2-2x+3$, $v_3 = x^2-2$, $v_4 = x-5$

ax^2+bx+c

Does $\{v_1, v_2, v_3, v_4\}$ span $\mathbb{P}_3$? Yes ✓

$$r_1v_1 + r_2v_2 + r_3v_3 + r_4v_4 = ax^2+bx+c$$

$$r_1(1+x) + r_2(x^2-2x+3) + r_3(x^2-2) + r_4(x-5) = ax^2+bx+c$$

$$\begin{aligned} r_2+r_3 &= a \\ r_1-2r_2+r_4 &= b \\ r_1+3r_2-2r_3-5r_4 &= c \end{aligned}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & a \\ 1 & -2 & 0 & 1 & b \\ 1 & 3 & -2 & -5 & c \end{array}\right] \xrightarrow[\text{then } -r_1+r_3 \to r_3]{r_1 \leftrightarrow r_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & b \\ 0 & 1 & 1 & 0 & a \\ 0 & 5 & -2 & -6 & c-b \end{array}\right]$$

$$\xrightarrow{-5r_2+r_3\to r_3}\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & b \\ 0 & 1 & 1 & 0 & a \\ 0 & 0 & -7 & -6 & c-b-5a \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}r_3\to r_3}\begin{array}{c} \begin{array}{ccccc} r_1 & r_2 & r_3 & r_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & b \\ 0 & 1 & 1 & 0 & a \\ 0 & 0 & 1 & 6/7 & (5a+b-c)/7 \end{array}\right] \end{array}$$

REF ✓ $\quad r_4 =$ free $\quad$ inf. many soln. ✓

Ex $\quad \mathbb{P}_3 \qquad v_1 = 1 + x^2, \quad v_2 = 3, \quad v_3 = x^2 - 5$

Does $\{v_1, v_2, v_3\}$ span $\mathbb{P}_3$? No.

$$r_1 v_1 + r_2 v_2 + r_3 v_3 = ax^2 + bx + c$$

$$r_1(1+x^2) + r_2(3) + r_3(x^2-5) = ax^2 + bx + c$$

$$\begin{aligned} r_1 + r_3 &= a \\ 0 &= b \\ r_1 + 3r_2 - 5r_3 &= c \end{aligned}$$

$\dashrightarrow$ If $b \neq 0 \Rightarrow$ no soln.

$\to$ a restriction for ax^2+bx+c

Linear Independence

vectors $\quad \star, \square, \triangle \in V$

$$c_1 \cdot \star + c_2 \square + c_3 \triangle = 0_V$$

$\to$ we will try to solve a homogeneous system of linear eqns.

If $c_1 = c_2 = c_3 = 0$ is the only soln. here $\iff \star, \square, \triangle$ are linearly independent.

$\{\star, \square, \triangle\} \to$ This set is linearly independent.

D $\quad \{v_1, v_2, \ldots, v_n\}$ is "linearly independent"

! $\{v_1, v_2, \dots, v_n\}$ is "linearly independent"

iff $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0_v$ has only the trivial solution:

$c_1 = c_2 = \dots = c_n = 0$

(we don't want inf. many soln. case).

! $\{v_1, v_2, \dots, v_m\}$ if this set includes $\vec{0}_v$ $\Rightarrow$ the set can not be linearly independent.

Ex $v_4 = \vec{0}_v$

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + \dots + c_mv_m = 0$$

$v_4 \to \vec{0}_v$

$c_4 \in \mathbb{R}$ $\Rightarrow$ leads to inf many soln case.

! $\{v_1, v_2, \dots, v_m\}$ if any vector in this set can be written as a linear combination of some other vectors in the set,

$\Rightarrow$ the set can not be linearly independent.

Ex $\vec{v}_5 = 3\vec{v}_1 + 2\vec{v}_2$

$$c_1v_1 + c_2v_2 + \dots + c_5v_5 + \dots + c_mv_m = \vec{0}_v$$

$v_5 \to 3v_1 + 2v_2$

$c_1 = -3c_5$, $c_2 = -2c_5$, $c_5 \in \mathbb{R}$ $= 0$ ✓ $c_3 = 0 \dots c_m = 0$

inf. many soln.

Ex $\mathbb{R}^3$ $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ Is $\{v_1, v_2\}$ lin. independent?

$$c_1v_1 + c_2v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{aligned} 1c_1 + 4c_2 &= 0 \\ 2c_1 + 5c_2 &= 0 \\ 3c_1 + 6c_2 &= 0 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned} ✓$$

$\Rightarrow \{v_1, v_2\}$ is lin. independent.

Ex/ $\mathbb{R}^3$ $v_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$ Is $\{v_1, v_2\}$ lin. independent?

No!

$v_2 = -2\vec{v}_1$

$c_1 v_1 + c_2 v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 3 & -6 & 0 \end{array}\right]$

$\rightarrow \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $c_2 =$ free, $c_1 = \ldots$

inf. many soln.

Ex/ $\mathbb{R}^3$ $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ $v_4 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Is $\{v_1, v_2, v_3, v_4\}$ lin. independent? ✗

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 3 & 0 \\ 3 & 1 & 4 & 5 & 0 \end{array}\right]$$

REF $\rightarrow \begin{bmatrix} 1 & & & c_4 \\ & 1 & & \\ & & 1 & \end{bmatrix}$ at least one free variable should come here

$\Rightarrow$ inf. many soln.

For $\mathbb{R}^n$ $> n$ vectors $\Rightarrow$ the set can not be lin. independent.

$\leq n$ vectors $\Rightarrow$ may or may not be.

($= n$ make use of $\boxed{\det A \neq 0}$ $\Rightarrow$ trivial soln ✓

$\det(A) = 0 \Rightarrow$ not lin. indep.)

For $\mathbb{P}_n$ " $\Rightarrow$

For $\mathbb{R}^{m \times n}$ $> mn$ vectors $\Rightarrow$ thes

Minimum Spanning Set for V

$\dim(\mathbb{R}^n)=n$ $V = \underline{\mathbb{R}^n}$ $= \underline{n}$ vectors

$\dim(\mathbb{P}_n)=n$ $V = \underline{\mathbb{P}_n}$ $= \underline{n}$ vector

$\dim(\mathbb{R}^{m\times n})=mn$ $V = \underline{\mathbb{R}^{m\times n}}$ $= mn$ vectors

linear independence ✓ + span ✓

$\det(A) \neq 0$ ✓

$\Rightarrow$ The set $\{v_1, \dots, \}$ is a basis for V

$\downarrow$

⇝ Span ✓

⇝ linear indepence ✓

$\Rightarrow$ "Basis"

\# elements in a basis of V = dimension of V

$= \dim(V)$